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ROBUST, ADAPTIVE, AND NETWORK CONTROL

Adaptive Output Control with the Given Control Quality Guaranteed

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Abstract—An algorithm for output control of linear plants with an arbitrary relative degree under the conditions of parametric uncertainty and bounded perturbations is proposed. Unlike classical adaptive control algorithms, the proposed algorithm allows ensuring the plants output tracking of the reference signal, with the tracking error being in the set specified by the developer. An example illustrating the efficiency of the proposed method is given.

Keywords: adaptive control, dynamic system, change of coordinates, stability, control

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1. INTRODUCTION

In this work, we consider the adaptive output tracking problem with respect to the reference signal with the given control quality guaranteed at any instant. We consider linear systems under parametric uncertainty and external perturbations. First stated in the 1950s, the model reference adaptive control problem is among the most studied ones. It is still relevant despite its respectable age. There are new problems arising, such as increasing the computational performance of algorithms, improving the quality of regulator tuning, generalizing the methods to include wider classes of systems, etc. The first solutions of adaptive control problems were associated with a number of assumptions, such as measurability of the state vector, the assumption for the systems transfer function to be strictly positive and real, or the master signal or external perturbations to be reproducible using an autonomous generator [1–4]. Approaches based on the apparatus of the second Lyapunov method [5] and hyperstability theory [6] required measuring higher derivatives of the tracking error. Later, this problem was solved using the augmented error method [7], high-order adaptation algorithms [8], perturbation compensation methods [9, 10], etc. One of the key issues faced by the above methods is that there is no way to influence the quality of the transient processes [11]. Various options have been proposed to partially resolve this, such as the accelerated convergence scheme [12, 13], the scheme with big coefficients in feedback [14], the nonvanishing excitation condition met [15], non-smooth control laws [16], etc. Thus, the methods [7–10, 12–16] impose significant constraints and solve the problem incompletely, with the control objective attained only in asymptotics. At the same time, the obtained estimates of the characteristics of the limit set are sufficiently rough. In [17–21], a method is proposed that ensures the output signal is in the given set. In [22], based on the method from [17–21], an adaptive control law with the output signal belonging to the given set is proposed. However, in [22], the relative degree of the plant is assumed to equal the unity. In this work, using the approaches from [17–22] and a modified adaptation algorithm [23], we propose a new solution to the adaptive control problem with the given control quality guaranteed [7, 8, 12, 14–16] for minimum-phase plants with an arbitrary relative degree. The structure of the work is as follows. In Section 2, we state the adaptive tracking problem with constraints on the output variable. In Section 3, we first synthesize a control law that assumes measurability of the derivatives of the plants output signal. The obtained solution is then generalized to the case when these derivatives are not measurable. In Section 4, we give numerical simulations to illustrate the efficiency of the obtained solution.

2. STATEMENT OF THE PROBLEM

We consider the dynamic system

$$Q(p)y(t) = kR(p)u(t) + f(t), \tag{1}$$

where $t \geqslant 0$, $u(t) \in \mathbb{R}$ is the control signal, $y(t) \in \mathbb{R}$ is the measurable controlled signal, $f(t) \in \mathbb{R}$ is the bounded external perturbation, $Q(s)$ and $R(s)$ are normalized polynomials (i.e., polynomials with higher coefficients equaling the unity) with unknown real coefficients and with the degrees equaling n and m, respectively, $\rho = n - m \geqslant 1$, the polynomial $R(s)$ is Hurwitz, $p = d/dt$ is the differentiation operator, the coefficient $k > 0$ is unknown, the boundary conditions $y^{(i)}(0),\ i = \overline{2, n}$ are unknown, but the set of initial conditions $y(0)$ is known. Throughout this work, s is a complex variable.

We give the reference model

$$T(p)y_m(t) = k_m g(t), \tag{2}$$

where $g(t) \in \mathbb{R}$ is a bounded and $(\rho - 1)$ times differentiable master control, $y_m(t) \in \mathbb{R}$ is the output of the reference model, $T(s)$ is the known normalized Hurwitz polynomial with real coefficients and the degree ρ, $k_m > 0$.

The purpose of this work is to synthesize a control law that ensures the tracking error $e(t) = y(t) - y_m(t)$ belongs to the set

$$\mathcal{E} = \left\{ (t, e) \in \mathbb{R}^2 \mid t \geqslant 0,\ \underline{g}(t) < e(t) < \overline{g}(t) \right\}, \tag{3}$$

where the functions $\underline{g}(t) < 0$, $\overline{g}(t) > 0$, $\overline{g}(t) - \underline{g}(t) > \delta$, $\delta > 0$ are bounded and have bounded first derivatives for any $t \geqslant 0$, and $e(0) \in \mathcal{E}$. The derivatives $\underline{g}(t)$ and $\overline{g}(t)$ should be bounded for the method to be applied [19].

3. PRINCIPAL RESULT

3.1. Synthesizing an Ideal Control Law

We introduce an auxiliary control v and first assume its derivatives to be measurable. We consider the control law

$$u(t) = \frac{T(p)}{p} v(t). \tag{4}$$

We represent the polynomials $Q(s)$ and $R(s)$ as $Q(s) = Q_m(s) + \Delta Q(s)$ and $R(s) = R_m(s) + \Delta R(s)$, where the normalized Hurwitz polynomials $Q_m(s)$ and $R_m(s)$ have the degrees n and m, respectively, and $Q_m(s)/R_m(s) = T(s)$. The unknown polynomials $\Delta Q(s)$ and $\Delta R(s)$ have degrees $n - 1$ and $m - 1$, respectively. Taking into account (4), we transform (1) to the form

$$Q_m(p)y(t) = \frac{kR_m(p)T(p)}{p} v(t) + \frac{k\Delta R(p)T(p)}{p} v(t) - \Delta Q(p)y(t) + f(t). \tag{5}$$

Dividing (5) by $Q_m(p)$, $kR_m(p)T(p)$ and p, we write the result as

$$y(t) = \frac{k}{p} \left[v(t) + \frac{\Delta R(p)}{R_m(p)} v(t) - \frac{p\Delta Q(p)}{kQ_m(p)} y(t) + \frac{p}{kQ_m(p)} f(t) + \epsilon_1(t) \right], \tag{6}$$

where $\epsilon_1(t)$ is an exponentially damped function depending on initial conditions (1). We express $y_m(t)$ from (2) as

$$y_m(t) = \frac{k}{p}\left[\frac{k_m}{k} g_r(t) + \epsilon_2(t)\right], \tag{7}$$

where $g_r(t) = \frac{p}{T(p)} g(t)$, $\epsilon_2(t)$ is an exponentially damped function depending on initial conditions (2). Taking into account the structure of (6) and (7), we rewrite $e(t)$ as

$$e(t) = \frac{k}{p}\left[v(t) - c_{01}y(t) - c_{02}^\top \zeta_y(t) - c_{03}^\top \zeta_v(t) - \frac{k_m}{k} g_r(t) + \frac{p}{kQ_m(p)} f(t) + \epsilon(t)\right], \tag{8}$$

where $\epsilon(t) = \epsilon_1(t) - \epsilon_2(t)$, the signals $\zeta_y(t)$, $\zeta_v(t)$ and $g_r(t)$ are formed using the following filters

$$\begin{aligned}
&\dot\zeta_y(t) = F_y\zeta_y(t) + b_y y(t), \quad \zeta_y(0) = 0,\\
&\dot\zeta_v(t) = F_v\zeta_v(t) + b_v v(t), \quad \zeta_v(0) = 0,\\
&\dot\zeta_g(t) = F_g\zeta_g(t) + b_g g(t), \quad \zeta_g(0) = 0, \quad g_r(t) = L_2\zeta_g(t).
\end{aligned} \tag{9}$$

Here, $b_i^\top = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}$ is the vector-column with the unity at the last position and zeros at the others, $i \in \{y, v, g, \eta\}$, $L_j = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}$ is the row vector with the unity at the jth position and zeros at the others. Here and in what follows, the matrices b_i and L_j have the same structure, and their dimensionality is clear from the context. The matrices F_y, F_v and F_g of filters (9) are given in the Frobenius form with the characteristic polynomials $Q_m(s)$, $R_m(s)$ and $T(s)$, respectively. The coefficient c_{01} is obtained from the expression

$$\frac{p\Delta Q(p)}{kQ_m(p)} = c_{01} + \frac{\Delta\tilde{Q}(p)}{kQ_m(p)},$$

where $\Delta\tilde{Q}(s)$ is an unknown polynomial of the degree $n-1$. The vectors c_{02} and c_{03} consist of the coefficients of the polynomials $\Delta\tilde{Q}(s)/k$ and $\Delta R(s)/k$, respectively. We introduce the vector of constant unknown parameters $c_0^\top = \begin{bmatrix} c_{01} & c_{02}^\top & c_{03}^\top & k_m/k \end{bmatrix}$ and the regression vector $\omega^\top(t) = \begin{bmatrix} y(t) & \zeta_y^\top(t) & \zeta_v^\top(t) & g_r(t) \end{bmatrix}$. We rewrite (8) as

$$\dot e(t) = k\left(v(t) - c_0^\top\omega(t) + \bar f(t) + \epsilon(t)\right), \tag{10}$$

where $\bar f(t) = \frac{p}{kQ_m(p)} f(t)$ is the signal bounded since $Q_m(s)$ is Hurwitz.

We apply the method from [19] to set the auxiliary control $v(t)$ that ensures the error $e(t)$ is in set (3). To do this, we introduce an auxiliary signal $\varepsilon(t)$ calculated by the formula

$$e(t) = \Phi(\varepsilon(t), t), \tag{11}$$

where the function $\Phi(\varepsilon(t), t)$ satisfies the conditions

(a) $\underline{g}(t) < \Phi(\varepsilon(t), t) < \overline{g}(t)$ for any $t \geqslant 0$ and $\varepsilon \in \mathbb{R}$;

(b) the function $\Phi(\varepsilon(t), t)$ is continuously differentiable on $\varepsilon(t)$ and t, and for any $e \in \mathcal{E}$ and $t \geqslant 0$, the following holds

$$\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t), t) \neq 0;$$

(c) the function $\frac{\partial\Phi}{\partial t}(\varepsilon(t), t)$ is bounded for any $\varepsilon \in \mathbb{R}$ and $t \geqslant 0$.

We give an example of transformation (11) in the form

$$\Phi(\varepsilon(t), t) = \frac{\overline{g}(t)\exp(\varepsilon(t)) + \underline{g}(t)}{\exp(\varepsilon(t)) + 1}. \tag{12}$$

For other examples of such functions (11) of coordinates, see [19].

Taking into account (11), we consider the total time derivative of $e(t)$ in the form

$$\dot{e}(t) = \frac{\partial \Phi}{\partial \varepsilon}(\varepsilon(t), t)\dot{\varepsilon}(t) + \frac{\partial \Phi}{\partial t}(\varepsilon(t), t).$$

Taking into account (10) and property (b), we express $e(t)$ as

$$\dot{\varepsilon}(t) = \left(\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon(t), t)\right)^{-1} \left(k(v(t) - c_0^\top \omega(t) + \bar{f}(t) + \epsilon(t)) - \frac{\partial \Phi}{\partial t}(\varepsilon(t), t)\right). \tag{13}$$

Now, we choose an auxiliary control action and an adaptation algorithm in the form

$$v(t) = -\text{sgn}\left\{\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon(t), t)\right\} \alpha \varepsilon(t) + c^\top(t)\omega(t), \tag{14}$$

$$\dot{c}(t) = -\left(\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon(t), t)\right)^{-1} \varepsilon(t)\omega(t) - \gamma c(t), \tag{15}$$

where $\alpha > 0$ and $\gamma > 0$, $c(t)$ is the vector of adjustable parameters. The transformation $\Phi(\varepsilon(t), t)$ is chosen beforehand, and the degree of $\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon(t), t)$ is constant due to property (b), so the value of the function sgn$\{\cdot\}$ in (14) is known. Substituting (14) into (13), we obtain the following expression

$$\dot{\varepsilon}(t) = \left(\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon(t), t)\right)^{-1} \left(-\text{sgn}\left\{\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon(t), t)\right\} k\alpha\varepsilon(t) + k(c(t) - c_0)^\top \omega(t) + \Xi(t)\right), \tag{16}$$

where $\Xi(t) = k(\bar{f}(t) + \epsilon(t)) - \frac{\partial \Phi}{\partial t}(\varepsilon(t), t)$ is a bounded function. We state a theorem and prove it in the Appendix.

Theorem 1. *Suppose the functions* $\underline{g}(t)$ *and* $\overline{g}(t)$ *satisfy the imposed requirements* (*see the paragraph below* (3)), *and properties* (a)–(c) *of transformation* (11) *hold. Then, for any* $\alpha > 0$ *and* $\gamma > 0$ *in closed system* (1), (2), (9), (15), (16), *control objective* (3) *is attained.*

Remark 1. In [22], the problem with the plant's relative degree equaling 1 and the known coefficient k is considered. Unlike the algorithm presented in [22], the procedure proposed in this work allows excluding the term containing $e(t)$ from the dynamics $\varepsilon(t)$, choose a simplified control law, and overcome the issue with the unknown k.

3.2. Synthesizing a Feasible Control Law

Control law (4) contains the derivatives of the auxiliary control v up to the $(\rho - 1)$th order. We introduce the estimate $\tilde{v}$ of auxiliary control (14). Then, the new control law looks like

$$u(t) = \frac{T(p)}{p}\tilde{v}(t), \tag{17}$$

$$\dot{\xi}(t) = G_0\xi(t) + D_0(\tilde{v}(t) - v(t)), \quad \tilde{v}(t) = \xi_1(t) = L_1\xi(t). \tag{18}$$

Here, $\xi(t) \in \mathbb{R}^\rho$ is the vector of signal estimates v and its derivatives,

$$G_0 = \begin{bmatrix} 0 & I_{\rho-1} \\ 0 & 0 \end{bmatrix} \qquad D_0^\top = \begin{bmatrix} -\frac{d_1}{\mu} & -\frac{d_2}{\mu^2} & \ldots & -\frac{d_\rho}{\mu^\rho} \end{bmatrix},$$

the numbers d_i, $i = \overline{1, \rho}$ are chosen so that the matrix $G = G_0 + DL_1$ is Hurwitz, $D^\top = \begin{bmatrix} d_1 & d_2 & \ldots & d_\rho \end{bmatrix}$, and $\mu > 0$ is a sufficiently small number. We introduce the vector

$$\eta(t) = \Gamma^{-1}(\xi(t) - \theta(t)), \quad \Gamma = \text{diag}\left\{\mu^{\rho-1}, \mu^{\rho-2}, \ldots, \mu, 1\right\}, \tag{19}$$

where $\theta^\top(t) = \begin{bmatrix} v(t) & \dot{v}(t) & \ldots & v^{(\rho-1)}(t) \end{bmatrix}$. Taking into account (19), we write

$$\Delta v(t) = \tilde{v}(t) - v(t) = \mu^{\rho-1}\eta_1(t) = \mu^{\rho-1}L_1\eta(t), \quad \tilde{v}(t) = v(t) + \mu^{\rho-1}L_1\eta(t). \tag{20}$$

Taking into account (17) and (20), we express $\dot{\varepsilon}$ as

$$\begin{aligned}\dot{\varepsilon}(t) = \left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}\Bigg(&-\operatorname{sgn}\left\{\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right\}k\alpha\varepsilon(t)\\ &+ k(c(t)-c_0)^\top\omega(t) + \mu^{\rho-1}kL_1\eta(t) + \Xi(t)\Bigg).\end{aligned} \tag{21}$$

Taking into account (18) and (19), we obtain

$$\mu\dot{\eta}(t) = G\eta(t) - \mu b_\eta v^{(\rho)}(t). \tag{22}$$

We state the principal result of the work.

Theorem 2. *Suppose the hypotheses of Theorem 1 hold. Then, there exists a number μ_0 such that control objective* (3) *is attained for $\mu \leqslant \mu_0$ in closed system* (1), (2), (9), (15), (21), (22).

Remark 2. Equation (22) is a singularly perturbed dynamical system. The analysis of such systems shows [24] that under certain conditions on the right-hand side of the system and for sufficiently small μ, the system has the same dissipativity domain and the same attraction domain as the system for $\mu = 0$, which is equivalent to using an ideal control law. As we showed in Theorem 1, such a system attains objective (3), so it will be sufficient to use one of the results of the theory of singularly perturbed systems to prove the theorem.

Theorem 2 shows the existence of a sufficiently small parameter μ_0 while the search for its actual value is an unsolved problem. The search for some quantitative characteristics, whose structure includes high-order observers, is not always possible and often remains in the qualitative form [28, 29]. Also, works [30, 31] point out that the iterative search for the value μ_0, for which the stability of the closed-loop system is achieved, can be carried out at the simulation stage.

Proof of Theorem 2. We rewrite (21) and (22) as

$$\begin{aligned}\dot{\varepsilon}(t) &= \left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}\Bigg(-\operatorname{sgn}\left\{\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right\}k\alpha\varepsilon(t)\\ &\qquad + k(c(t)-c_0)^\top\omega(t) + \mu_2^{\rho-1}kL_1\eta(t) + \Xi(t)\Bigg),\\ \mu_1\dot{\eta}(t) &= G\eta(t) - \mu_2 b_\eta v^{(\rho)}(t),\end{aligned} \tag{23}$$

where $\mu_1 = \mu_2 = \mu$. We use the lemma [29].

Lemma 1. *If the system is described by the equation $\dot{x} = f(x,\mu_1,\mu_2)$, $x \in \mathbb{R}^{m_0}$, where $f(x,\mu_1,\mu_2)$ is a continuous function Lipschitzian with respect to x and has a bounded closed dissipativity domain $\mu_2 = 0$ for $D = \{x \mid F(x) < C\}$, where $F(x)$ is a continuous piecewise smooth function positive definite in $\mathbb{R}^{m_0}$ (in the sense of* [26]*), then there exists $\mu_0 > 0$ such that for $\mu_2 \leqslant \mu_0$ the original system has the same dissipativity domain D if the relation*

$$\sup_{|\mu_1|\leqslant\bar{\mu}_1}\left\{\left\langle\frac{\partial F}{\partial x}(x), f(x,\mu_1,0)\right\rangle\Big|_{F(x)=C}\right\} \leqslant -C_1$$

holds when $\mu_2 = 0$ for some $C_1 > 0$ and $\bar{\mu}_1 > 0$.

Substituting $\mu_2 = 0$ into (23), we have the closed-loop system with the ideal control law obtained in the previous subsection. Theorem 1 shows that the solutions of the closed-loop system tend to a bounded set, and the additional equation $\mu_1\dot{\eta}(t) = G\eta(t)$ does not violate this condition since G is Hurwitz. Hence, for a sufficiently small μ, the system preserves the dissipativity domain. The theorem is proved.

4. SIMULATION

Example 1. We consider control plant (1) with the following linear differential operators

$$Q(p) = p^4 + 6p^3 - 3p^2 - p + 2 \quad \text{and} \quad kR(p) = p + 1.$$

The initial conditions are $p^3y(0) = p^2y(0) = py(0) = y(0) = 1$. The external perturbation is $f(t) = 2\sin(1.5t) + d(t)$, where $d(t) = sat\{\hat{d}(t)\}$, $sat\{\cdot\}$ is the saturation function, $\hat{d}(t)$ is white noise simulated in Matlab Simulink using the Band-Limited White Noise block with the Noise power parameter equaling 1 and the Sample time parameter equaling 0.1. Reference model (2)

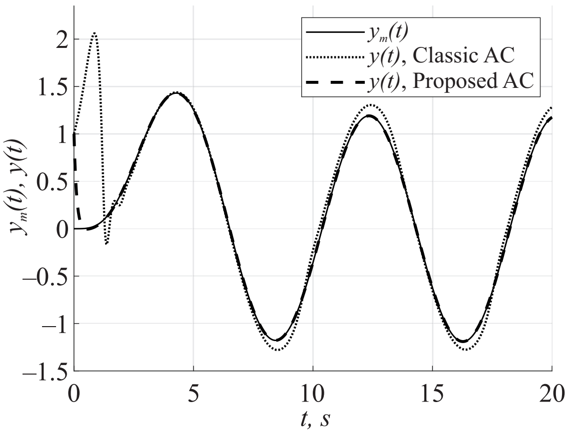

Fig. 1. The output of the reference model (the black curve), the output of plant (1) when the high-order adaptive robust algorithm is used (the blue curve), and the output of plant (1) when the proposed algorithm is used (the red curve).

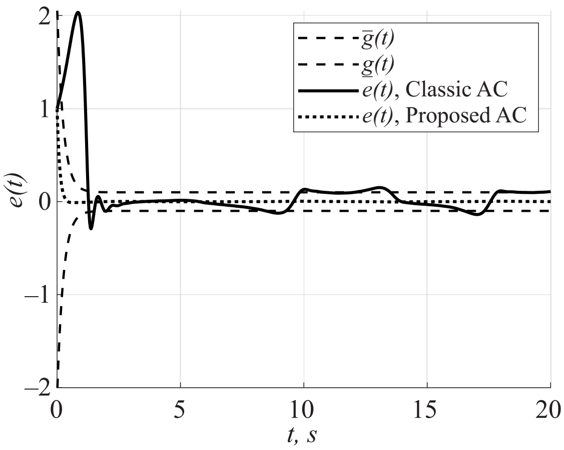

Fig. 2. The constraints $\overline{g}(t)$ and $\underline{g}(t)$ (the blue and red curves), the control error when the high-order adaptive robust algorithm is used (the green curve), and the control error when the proposed algorithm is used (the black curve).

that receives the master control $g(t) = 2.5\sin(0.8t)$ has the following parameters

$$T(p) = p^3 + 3p^2 + 3p + 1 \quad \text{and} \quad k_m = 1.$$

According to (4), we set the control law in the form $u(t) = \frac{T(p)}{p}v(t)$. As the function $\Phi(\varepsilon(t), t)$ we take (12). We give the constraints for the tracking error by the functions

$$\overline{g}(t) = 2\exp(-4t) + 0.1,$$
$$\underline{g}(t) = -\overline{g}(t).$$

We choose $Q_m(p) = p^4 + 4p^3 + 6p^2 + 4p + 1$, $R_m(p) = p + 1$ and set in filters (9):

$$F_v = -1, \quad F_g = \begin{bmatrix} 0 & I_2 & \\ -1 & -3 & -3 \end{bmatrix}, \quad F_y = \begin{bmatrix} 0 & I_3 & & \\ -1 & -4 & -6 & -4 \end{bmatrix}.$$

We choose the parameters $\alpha = 5$, $\gamma = 1$ and $\mu = 10^{-3}$ in (14), (15), and (18). We compare the proposed control algorithm with the classical high-order adaptive algorithm [31] with the parameters $\sigma_c = 5$, $\gamma_c = 100$ and $\mu_c = 200$.

Figures 1 and 2 show that the classical algorithm fails. First, there is no way to set the control quality during transient processes. Secondly, the limit set is not defined in advance and turns out to be bigger than that specified using the proposed algorithm.

5. CONCLUSIONS

In this work, we propose a new model reference adaptive tracking algorithm with the given control quality guaranteed over the entire interval of system operation. The method combines the idea of adaptive control [23] and the approach [19] used to synthesize a nonlinear control law under constraints. We compared the obtained result with the classical high-order adaptive control law [27] by simulation. The proposed algorithm allowed us to ensure that the tracking error is in a predetermined set at any instant while the methods from [23, 27] do not allow controlling the values of the tracking error in the transient mode. Moreover, the estimates of the value of the limit objective set obtained in [23, 27] depend on the values of unknown parameters and have overestimated values, therefore, they cannot be used to set the control accuracy in the steady-state mode. Thus, the proposed approach solves both these problems, and the control quality is completely determined by the choice of bounding functions.

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APPENDIX

Proof of Theorem 1. By property (b) of transformation (11), the function $\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t), t)$ is sign-definite. First, suppose that $\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t), t) > 0$. We choose a Lyapunov function of the form

$$V_1 = 0.5\varepsilon^2(t) + 0.5k(c(t) - c_0)^\top(c(t) - c_0) + \chi \int\limits_t^\infty \left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t), t)\right)^{-1} \epsilon^2(s)\, ds,$$

where $\chi > 0$. Taking into account (15) and (16), we find $\dot{V}_1$ in the form

$$\begin{aligned}\dot{V}_1 = -\alpha k\left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}\varepsilon^2(t) + \left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}\varepsilon(t)(\psi(t)+k\epsilon(t))\\ -\gamma k(c(t)-c_0)^\top c(t) - \chi\left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}\epsilon^2(t),\end{aligned} \tag{A.1}$$

where $\psi(t) = k\bar{f}(t) - \frac{\partial\Phi}{\partial t}(\varepsilon(t),t)$ is bounded. We use the following relations

$$\begin{aligned}\varepsilon(t)\psi(t) &\leqslant 0.5(\nu^{-1}\varepsilon^2(t)+\nu\psi^2(t)),\\ \varepsilon(t)\epsilon(t) &\leqslant 0.5(\nu^{-1}\varepsilon^2(t)+\nu\epsilon^2(t)),\\ -(c(t)-c_0)^\top c(t) &= -0.5((c(t)-c_0)^\top(c(t)-c_0)+c^\top(t)c(t)-c_0^\top c_0).\end{aligned} \tag{A.2}$$

Taking into account (A.2), we rewrite (A.1) as

$$\begin{aligned}\dot{V}_1 \leqslant& -\left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}(\alpha k - 0.5\nu^{-1}(1+k))\varepsilon^2(t)\\ &+0.5\left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}\nu\psi^2(t) + 0.5\gamma k c_0^\top c_0\\ &-0.5\gamma k((c(t)-c_0)^\top(c(t)-c_0)+c^\top(t)c(t))\\ &-\left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}(\chi-0.5k\nu)\epsilon^2(t).\end{aligned} \tag{A.3}$$

Note that it follows from properties (a) and (b) of function (11) that it is differentiable everywhere, strictly monotone and bounded, and hence $\sup_{t\geqslant 0}\left\{\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right\} < \infty$ is bounded. Moreover, $\sup_{t\geqslant 0}\{\psi(t)\} < \infty$ is bounded by property (c). Then, it follows from (A.3) that $\dot{V}_1 < 0$ is attained when the condition

$$\begin{aligned}&\nu > 0.5\alpha^{-1}(k^{-1}+1), \quad \chi > 0.5k\nu,\\ &|\varepsilon(t)| > \sqrt{\frac{0.5}{\alpha k - 0.5\nu^{-1}(1+k)}\left(\frac{\nu}{k}\sup_{t\geqslant 0}\{\psi(t)\}^2 + \sup_{t\geqslant 0}\left\{\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right\}\gamma k c_0^\top c_0\right)}\end{aligned} \tag{A.4}$$

holds.

Obviously, there will always be ν and χ such that (A.4) holds. Hence, $\dot{V}_1 < 0$.

Now, we consider the case $\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t) < 0$. We choose a Lyapunov function of the form

$$V_2 = 0.5\varepsilon^2(t) + 0.5k(c(t)-c_0)^\top(c(t)-c_0).$$

We find $\dot{V}_2$ as

$$\dot{V}_2 = \left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}k\alpha\varepsilon^2(t) + \left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}\varepsilon(t)(\psi(t)+k\epsilon(t)) - \gamma k(c(t)-c_0)^\top c(t). \tag{A.5}$$

Using relations (A.2) for $\nu = 1$, we rewrite (A.5) as

$$\begin{aligned}\dot{V}_2 \leqslant \left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}(\alpha k + 0.5k + 0.5)\varepsilon^2(t) + 0.5\left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}\psi^2(t) + 0.5\gamma k c_0^\top c_0\\ -0.5\gamma k((c(t)-c_0)^\top(c(t)-c_0)+c^\top(t)c(t)) + 0.5\left(\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right)^{-1}k\epsilon^2(t).\end{aligned} \tag{A.6}$$

Since $\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t) < 0$, all terms but $0.5\gamma k c_0^\top c_0$ are negative. Previously, we have shown that $\sup_{t\geqslant 0}\left\{\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right\} < \infty$. Then, it follows from (A.6) that $\dot{V}_2 < 0$ is attained when the condition

$$|\varepsilon(t)| > \sqrt{-\inf_{t\geqslant 0}\left\{\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon(t),t)\right\}\frac{0.5\gamma k c_0^\top c_0}{\alpha k + 0.5k + 0.5}} \tag{A.7}$$

holds.

Inequalities (A.4) and (A.7) define the sets trajectories (16) tend to in each of the cases involved. Then, by Theorem 3.1 from [19], trajectories (10) will belong to some subset $\mathcal{E}$, which means that objective (3) is fulfilled. The theorem is proved.

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